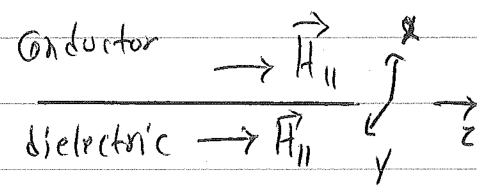
Lecture 22 04/11/2018

Waveguides

Electromagnetic waves can be made to propagate in confined channels that are hollow or filled with a dielectric material, and have either conducting walls or a dielectric material (called "cladding") surrounding them. In the latter case, the cladding must have a lower dielectric constant in order to confine the wave within the channel.

In both cases, confinement of the wave is due to reflection at the walls with little or no energy lost due to loss in materials or to transmission out of the channel.

First, let us consider loss in a conductor in the excellent conductor limit. In this case:



$$\vec{\nabla} \times \vec{H} \simeq \vec{J} = \sigma \vec{E}$$

$$\vec{E}_{\parallel} = \frac{1}{\sigma} \vec{J}_{\parallel} \simeq \frac{1}{\sigma} (\vec{\nabla} \times \vec{H})_{\parallel} \simeq \frac{1}{\sigma} (\vec{\nabla} \times \vec{H}_{\parallel})$$

Writing:

$$\vec{H}_{\parallel} = \hat{z} H_0 e^{i(kx-\omega t)}$$

We have:

$$k = \frac{\omega}{c}\sqrt{\frac{i\sigma}{\omega\epsilon_0}} = \frac{1}{\delta}(1+i) \qquad (\delta:\ \text{skin depth})$$

Thus:

$$\vec{H}_{\parallel} = \hat{z} H_0 e^{\frac{i}{\delta}(1+i)x - i\omega t} = \hat{y} H_{\parallel}$$

This implies that:

$$\vec{E}_{\parallel} = \frac{1}{\sigma}\vec{\nabla}\times\vec{H}_{\parallel} = -\frac{i}{\delta}(1+i) H_{\parallel}\hat{x} = \frac{1-i}{\sigma\delta} H_{\parallel}\hat{y}$$

Time-averaged power flux per unit area into the conductor is then given by:

$$\boxed{\frac{dP_{loss}}{da} = \frac{1}{2} Re(\vec{E}_{\parallel}\times\vec{H}_{\parallel}^{*}) = \frac{1}{2\sigma\delta}|H_{\parallel}|^2}$$

We note that, as expected, this is the same as $\int \frac{1}{2} Re(\vec{J}\cdot\vec{E}^{*})\,dx$.

To see this, we use:

$$\vec{E}_{\parallel} = \hat{y} E_{\parallel} e^{\frac{i}{\delta}(1+i)x - i\omega t} \Rightarrow Re(\vec{J}\cdot\vec{E}^{*}) = \sigma|\vec{E}_{\parallel}|^2 = \sigma|E_{\parallel}|^2 e^{-\frac{2x}{\delta}}$$

Hence:

$$\frac{1}{2}\int_0^\infty Re(\vec{J}\cdot\vec{E}^*)\,dx = \frac{\sigma\delta}{4}\cdot|E_{\parallel}|^2 = \frac{\cancel{\sigma}\delta}{2\cancel{4}}\frac{\cancel{2}}{\sigma^{\cancel{2}}\delta^{\cancel{2}}}|H_{\parallel}|^2 = \frac{1}{2\sigma\delta}|H_{\parallel}|^2$$

This confirms that the Poynting vector component into the conductor is the same as the power loss per area due to Joule heating.

Next, we consider loss in the dielectric medium of the guide. For the propagating wave, we have:

$$\vec{E}(\vec{x}) = \vec{E}(x,y)\,e^{i(kz-\omega t)}$$

Where:

$$k = \sqrt{\omega^2\epsilon\mu - \gamma^2}$$

For a loss dielectric, $\epsilon = \epsilon' + i\epsilon''$, where ϵ' and ϵ'' are real and positive.

Then:

$$k = k' + ik'' \Rightarrow \vec{H}(\vec{x}) = \vec{H}(x,y)\,e^{ikz - i\omega t}$$

And:

$$\frac{dP^{(d)}_{loss}}{dz} = \lim_{\Delta z\to 0}\frac{P(z) - P(z+\Delta z)}{\Delta z} = \frac{-dP}{dz}$$

Where:

$$P_{(z)} = \iint \frac{1}{2} Re(\vec{E} \times \vec{H}^*) \cdot \hat{z}\, dx\,dy = \frac{1}{2} Re \iint (\vec{E} \times \vec{H}^*) \cdot \hat{z}\, e^{-2k''z}\, dx\,dy$$

Thus:

$$\frac{dP^{(d)}_{loss}}{dz} = 2k'' P$$

The total loss then is:

$$\frac{dP^{(d)}_{loss}}{dz} + \frac{dP^{(c)}_{loss}}{dz}$$

Where:

$$\frac{dP^{(c)}_{loss}}{dz} = \frac{1}{dz} \oint_C \frac{dP_{loss}}{da}\, dz\, d\ell = \oint_C \frac{dP_{loss}}{da}\, d\ell = \frac{1}{2\sigma\delta} \oint_C |H_{\parallel}|^2\, d\ell$$

Here C is the cross-sectional boundary curve. Therefore, the total loss is:

$$\frac{1}{2\sigma\delta} \oint_C |H_{\parallel}|^2 d\ell + 2k'' P$$

Cylindrical Waveguides

We consider monochromatic wave ($\vec{E}, \vec{B} \propto e^{-i\omega t}$) in a waveguide whose cross-section is invariant along the direction of propagation.

However, we would like to note that the cross-sectional shape is in general arbitrary. The invariance of the boundary conditions with the longitudinal coordinate allows us to have solutions of the form:

$\vec{E}, \vec{B} \propto e^{i(kz-\omega t)}$

The Maxwell equations are:

$\vec{\nabla} \times \vec{E} = i\omega \vec{B}$, $\vec{\nabla} \times \vec{B} = -i\mu\epsilon\omega \vec{E}$, $\vec{\nabla} \cdot \vec{E} = 0$, $\vec{\nabla} \cdot \vec{B} = 0$

We consider three general types of modes that are supported by a cylindrical waveguide:

(a) TEM modes. These are modes for which $\vec{E}$ and $\vec{B}$ are transverse (i.e., $\vec{E}, \vec{B} \perp \hat{z}$). Hence:

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{E}_T = \vec{\nabla}_T \times \vec{E}_T + \hat{z} \frac{\partial \vec{E}_T}{\partial z} = \vec{\nabla}_T \times \vec{E}_T + ik\hat{z} \times \vec{E}_T = i\omega \vec{B}_T$$

This implies that:

$$\boxed{\vec{\nabla}_T \times \vec{E}_T = 0} \quad , \quad \boxed{\vec{B}_T = k\hat{z} \times \vec{E}_T}$$

Similarly:

$$\vec{\nabla}\cdot\vec{E}=0 \Rightarrow \boxed{\vec{\nabla}_T\cdot\vec{E}_T=0}$$

The two equations $\vec{\nabla}_T\times\vec{E}_T=0$ and $\vec{\nabla}_T\cdot\vec{E}_T=0$ have an electrostatic character. Therefore, we can write:

$$\vec{E}_T=-\vec{\nabla}\Phi \Rightarrow \nabla^2\Phi=0$$

Since this equation has a unique solution for a given boundary condition, a single conducting wall cannot confine a TEM mode. The reason is that $\nabla^2\Phi=0$ with $\Phi_{wall}=$ Const. (for a conducting wall) gives rise to $\Phi=$ Const., and hence $E_T=B_T=0$, within the waveguide. However, TEM modes can propagate with more than one conducting wall, for example, a transmission line consisting of two concentric cylinders.

The wave equation for the TEM modes is:

$$(\nabla^2+\omega^2\epsilon\mu)\vec{E}_T=0 \Rightarrow \left(\nabla_T^2+\frac{\partial^2}{\partial z^2}+\omega^2\epsilon\mu\right)\vec{E}_T=0 \quad \Rightarrow$$

$$[\nabla_T^2 + (\omega^2 \epsilon \mu - k^2)] \vec{E}_T = 0$$

Since $\nabla_T^2 \vec{E}_T = \vec{\nabla}_T (\cancelto{0}{\vec{\nabla}_T \cdot \vec{E}_T}) - \vec{\nabla}_T \times (\cancelto{0}{\vec{\nabla}_T \times \vec{E}_T}) = 0$, we have:

$$\omega^2 \epsilon \mu - k^2 = 0 \Rightarrow \boxed{k = \omega \sqrt{\epsilon \mu}}$$

This is the same dispersion relation as a plane wave in a dielectric medium.